AVERAGED EQUATIONS FOR DEVELOPING FLOW OF A FLUID-SOLID MIXTURE

M. MASSOUDI*
U.S. Department of Energy
National Energy Technology Laboratory
P.O. Box 10940
Pittsburgh, PA 15236-0940, USA
e-mail: Massoudi@netl.doe.gov

A mathematical description of a mixture of a Newtonian fluid infused with particulate solids is presented within the context of Mixture Theory. In the absence of any thermal effects, the balance of mass and balance of linear momentum equations for each component are averaged over the cross section of the flow to obtain ordinary differential equations describing developing flow between parallel plates. The resulting coupled equations describe the variation of the average velocities and volume fraction in the direction of flow, and represent a simplified approximate set of equations which are used in engineering applications.

Key words: mixture theory, developing flows, multiphase flows, averaging method, granular materials.

1. Introduction

Flowing mixtures consisting of solid particles entrained in a fluid are relevant to a variety of applications such as fluidized beds and pneumatic transport of solid particles. The importance of these complex flows is discussed, for example, by Soo (1989, 1990a) and Marcus *et al.* (1990) who provide up-to-date accounts of multiphase fluid dynamics and pneumatic conveying of solids. Many of the articles published concerning fluid-solid flows typically employ one of two continuum theories developed to describe such situations: averaging or mixture theory (theory of interacting continua).

In the averaging approach (Anderson and Jackson, 1967; Drew and Segel, 1971) point-wise equations of motion, valid for a single fluid or a single particle, are modified to account for the presence of the other components and the interactions between components. These equations are then averaged over time,

* To whom correspondence should be addressed

some suitable volume which is large compared with a characteristic dimension (for example, particle spacing or the diameter of solid particles) but small compared to the dimensions of the whole system, or an ensemble. Terms which appear due to the process of averaging, which are not present in the equation being averaged, are usually interpreted as some form of interaction between the constituents. Constitutive relations to represent these interaction forces, as well as the stress tensors for each constituent, are required to make the theory complete. A comparison of recent formulations of multiphase flows is provided in two review articles by Soo (1990b, 1991).

The second method of modeling multi-component systems is the mixture theory. This theory, which traces its origins to the work of Fick (1855), was first presented within the framework of continuum mechanics by Truesdell (1957). It is a means of generalizing the equations and principles of the mechanics of a single continuum to include any number of superimposed continua. The fundamental assumption of the theory is that at any instant of time, every point in space is occupied by one particle from each constituent, in a homogenized sense. Like the averaging, the mixture theory also requires constitutive relations for the stress tensor of each component of the mixture and for momentum exchange between the components.

Recently Rajagopal, Massoudi and co-workers have proposed a mathematical description for a flowing mixture of solid particles and a fluid within the context of the mixture theory (Johnson *et al.*, 1991a, b; Massoudi *et al.*, 1999). The mixture is modeled as a two-component mixture of a Newtonian fluid and a granular solid, in a manner that the equations reduce to those describing a linearly viscous fluid when the solid volume fraction goes to zero, and to those describing a flowing granular solid when the fluid volume fraction goes to zero. Boundary value problems have been solved numerically for a steady, fully developed flow of this mixture between parallel plates, through a pipe, and between two rotating cylinders (Johnson *et al.*, 1991a, b; Massoudi and Johnson, 2000).

From a practical point of view, a dense suspension of solid particles in a fluid shows different characteristics for different suspending media. For example, Soo (1987) shows that a steady flow in a dense gas-solid suspension is not expected; the flow is often turbulent. Soo (1984) also indicates that "the minimum suspension velocity of the same solid particles and the pressure drop are much lower in a liquid than in a gas at similar temperature and at useful working pressures". Also, gas-solid mixtures have become increasingly important in many of the chemical processes and energy related technologies such as pneumatic transport, flow of pulverized coal in feeder lines to surfaces, and fluidized beds. In suspensions of gas-solid flows, particle-particle interaction has also received much attention (Soo, 1967).

Cross-sectional or radial variations of flow properties, such as velocity and density, are very difficult to measure in gas-solid flows. In many cases the measurements are restricted to cross-sectionally averaged quantities. A review of measurement techniques for two-phase media is given by Hewitt in the book by Hestroni (1982). The purpose of this paper is to apply the mixture theory to a steady developing flow, and by averaging the equations over the cross section of the flow derive equations governing the average volume fraction and the average velocities of each component. First, we review briefly the basic principles of mixture theory and discuss constitutive equations for the mixture components and for the interactions between components. We then average the balance of mass and balance of linear momentum equations over an appropriate control volume.

The main objective of this work is to obtain one-dimensional equations of motion for a steady developing flow of a mixture. For many practical applications in gas suspension flows, or liquid suspensions, generally a one-dimensional equation describing the overall momentum equation for the mixture is given (see Soo, 1967, p.279, Eq.(7.7)). In these studies, the particles are treated as a dilute phase. In a previous study, Massoudi *et al.* (1999) provided the mixture momentum equation for a fully developed flow of a dense mixture in a pipe. In this paper, we will use the same constitutive relations to derive the appropriate form of the one-dimensional steady developing flow of such a mixture.

2. Mixture theory

2.1. Introduction

Materials such as steel, water, or rubber are usually regarded as a single continuum. In many applications such as a fluid containing particles, however, it is useful to describe the two components as separate, interacting continua. A general mixture theory, or theory of interacting continua, can be used to derive balance equations for any number of continuous bodies occupying the same space. The details and historical development of the mixture theory are found in review articles by Atkin and Craine (1976), Bedford and Drumheller (1983), Bowen (1976), several appendices in the recent edition of *Rational Thermodynamics* by Truesdell (1984), and the book by Rajagopal and Tao (1995).

2.2. Kinematics and notation

The underlying assumption of the mixture theory is that the mixture may be regarded as n superimposed continua, each having its own motion. At any time t, each position in the mixture is occupied by one particle from each constituent of the mixture. As in the case of a single continuum, each constituent of the mixture is assigned an arbitrary fixed reference configuration. The motion of a particle of constituent α is a one-to-one, invertible mapping denoted by

$$x_\alpha = \chi_\alpha(X_\alpha, t) \tag{2.1}$$

where X_α is the position of a particle of the α-th body or constituent in its reference configuration, t the time, and x_α the spatial position occupied at time t by the particle that was at X_α in the reference configuration. In general, sufficient smoothness is assumed in order to make any needed mathematical operations correct. The velocity vectors corresponding to the motions are

$$v_\alpha = \frac{D_\alpha \chi_\alpha}{Dt}, \tag{2.2}$$

$\frac{D_\alpha}{Dt}$ denotes differentiation with respect to t, holding X_α fixed. Note that there is no sum on α. The densities of each component of the mixture, measured per unit volume of the mixture, are written ρ_α. The mean velocity of the mixture V, is defined through

$$\rho V = \sum_{\alpha=1}^{n} \rho_\alpha v_\alpha \tag{2.3}$$

where ρ is the mixture density, defined by

$$\rho = \sum_{\alpha=1}^{n} \rho_\alpha . \tag{2.4}$$

Consider the special case of a two component mixture consisting of a Newtonian fluid and a granular material. The fluid in the mixture will be represented by S_1 and the granular solid by S_2. Let X_1 and X_2 denote the positions of the particles of S_1 and S_2 in the reference configuration. The motion of the constituents is represented by the mappings

$$x_1 = \chi_1(X_1, t) \quad \text{and} \quad x_2 = \chi_2(X_2, t) \tag{2.5}$$

where the subscripts 1 and 2 refer to the fluid and granular solid, respectively. The kinematical quantities associated with these motions are

$$v_1 = \frac{D_1 \chi_1}{Dt}, \qquad v_2 = \frac{D_2 \chi_2}{Dt}, \tag{2.6}$$

$$a_1 = \frac{D_1 v_1}{Dt}, \qquad a_2 = \frac{D_2 v_2}{Dt}, \tag{2.7}$$

$$L_1 = \frac{\partial v_1}{\partial x_1}, \qquad L_2 = \frac{\partial v_2}{\partial x_2}, \tag{2.8}$$

$$D_1 = \frac{1}{2}\left(L_1 + L_1^T\right), \qquad D_2 = \frac{1}{2}\left(L_2 + L_2^T\right), \tag{2.9}$$

$$W_1 = \frac{1}{2}\left(L_1 - L_1^T\right), \qquad W_2 = \frac{1}{2}\left(L_2 - L_2^T\right) \tag{2.10}$$

where v denotes velocity, a acceleration, L is the velocity gradient, D the symmetric part of the velocity gradient, and W the spin tensor.

Also, ρ_1 and ρ_2 are the densities of the mixture components in the current configuration given by

$$\rho_1 = \phi\rho_f, \qquad \rho_2 = \nu\rho_s \tag{2.11}$$

where ρ_f is the density of the pure fluid, ρ_s the density of the solid grains, and ν the volume fraction of the solid component and ϕ the volume fraction of the fluid. For a saturated mixture $\phi = 1 - \nu$. The mixture density, ρ_m is given by

$$\rho_m = \rho_1 + \rho_2, \tag{2.12}$$

and the mean velocity v of the mixture is defined by

$$\rho_m v = \rho_1 v_1 + \rho_2 v_2. \tag{2.13}$$

2.3. Basic equations

Balance equations for the mixture, and its constituents, may again be in either integral form or in differential form. In the absence of any thermal, electric, or magnetic effects, the conservation of mass for the fluid and granular material is

$$\frac{D}{Dt}\int_{P_t} \rho_1 dV = \int_{P_t} c_1 dV, \qquad \forall P_t \subseteq \Omega_t \tag{2.14}$$

and

$$\frac{D}{Dt}\int_{P_t}\rho_2 dV = \int_{P_t} c_2 dV, \qquad \forall P_t \subseteq \Omega_t \tag{2.15}$$

where c_1 and c_2 are the mass supplies to the first and second constituents, respectively. These equations take the local form

$$\frac{\partial \rho_1}{\partial t} + div\left(\rho_1 v_1\right) = c_1 \tag{2.16}$$

and

$$\frac{\partial \rho_2}{\partial t} + div\left(\rho_2 v_2\right) = c_2. \tag{2.17}$$

Let T_1 and T_2 denote the partial stress tensors of the fluid S_1 and the solid S_2, respectively. Then the balance of linear momentum for the fluid and solid are given by

$$\begin{aligned} \frac{D}{Dt}\int_{P_t}\rho_1 v_1 dV &= \int_{\partial P_t} T_1^T n_1 da + \\ &+ \int_{P_t}\left(\rho_1 b_1 + f_1 + c_1 v_1\right) dV, \qquad \forall P_t \subseteq \Omega_t \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \frac{D}{Dt}\int_{P_t}\rho_2 v_2 dV &= \int_{\partial P_t} T_2^T n_2 da + \\ &+ \int_{P_t}\left(\rho_2 b_2 - f_1 + c_2 v_2\right) dV, \qquad \forall P_t \subseteq \Omega_t \end{aligned} \tag{2.19}$$

or

$$\rho_1 \frac{D_1 v_1}{Dt} = div T_1^T + \rho_1 b_1 + f_1 + c_1 v_1 \tag{2.20}$$

and

$$\rho_2 \frac{D_2 v_2}{Dt} = div T_2^T + \rho_2 b_2 - f_1 + c_2 v_2 \tag{2.21}$$

where b represents the external body force, and f_1 represents the mechanical interaction (local exchange of momentum) between the components.

The balance of moment of momentum implies that

$$T_1 + T_2 = T_1^T + T_2^T . \tag{2.22}$$

The partial stresses need not by symmetric, however.

2.4. Boundary conditions

One difficulty in using the mixture theory is specifying the boundary conditions. Boundary conditions can be prescribed based on the tractions acting on the boundary, known displacements (or velocities) on the boundary, or some combination of the two.

The difficulty in specifying tractions is that one must ultimately determine how much of the total traction is supported by each constituent. Rajagopal *et al.* (1986), Rajagopal and Tao (1995) and Tao and Rajagopal (1995) have addressed this issue for a certain class of boundary value problems. The problems considered here belong to the second class in which the velocities are specified at the boundaries, for instance the adherence boundary condition (Massoudi *et al.*, 1999) or a slip condition (Massoudi and Phuoc, 2000) that is specified on the basis of experiments or observation.

3. Constitutive equations

We will give a brief description of the models that we intend to use. Fuller explanations are given in our other papers. In the absence of any thermal and chemical effects, for a purely mechanical system, the constitutive quantities which are to be modeled are the stress tensors and the interaction force. We assume that the fluid and solid phases are dense enough to be modeled as homogeneous continuous media. Based on our knowledge of modeling in the theory of granular materials, it would be natural to assume that all the constitutive functions depend on (Rajagopal *et al.*, 1990)

$$\rho_1,\ \rho_2,\ \nabla\rho_1,\ \nabla\rho_2, \nabla\nabla\rho_1,\ \nabla\nabla\rho_2,\ v_1 - v_2,\ D_1,\ D_2 \tag{3.1}$$

and possibly other vectors and tensors. Then restrictions can be obtained using the second law on the forms of the constitutive relations for the constituents (Shi *et al.*, 1981; Atkin and Craine, 1976). Here, we discuss an alternative approach, which is to postulate the constitutive expressions by simply generalizing the structure of the constitutive relations from a single constituent theory. In general, the constitutive expressions for T_f and T_s depend on the kinematical quantities associated with both the constituents. However, we assume that T_s and T_f depend only on the kinematical quantities associated with the solid and fluid, respectively. This assumption is sometimes called "the principle of phase separation" and was first used in the mixture theory by Adkins (1963a, b).

In the majority of fluid-solid mixtures, the fluid is either a gas or water. Therefore, it is appropriate to assume that the fluid behaves as a linearly viscous fluid, whose constitutive equation is

$$T_f = \left[-p(\rho_1) + \lambda_f(\rho_1)\, tr\, D_1\right] I + 2\mu_f(\rho_1) D_1 \tag{3.2}$$

where p is the fluid pressure, λ_f and μ_f are the viscosities, D_1 is the symmetric part of the velocity gradient for the fluid defined in Eq.(2.9) and I is the identity tensor. If the fluid is incompressible, then p is one of the unknown quantities in the problem that would have to be calculated. If the fluid is compressible, an equation of state is needed. In general p, λ_f, and μ_f are functions of ρ_1 and temperature.

There are basically two different ways of deriving a constitutive relation for the stress tensor of granular materials – the continuum approach and the statistical approach. We use the continuum approach in our analysis. In this study, we assume that the stress tensor for the granular materials is given by (Rajagopal and Massoudi, 1990)

$$\begin{aligned} T_s = &\left[\hat{\beta}_0(\rho_2) + \hat{\beta}_1(\rho_2)\,\mathrm{grad}\,\rho_2 \cdot \mathrm{grad}\,\rho_2 + \hat{\beta}_2(\rho_2)\, tr\, D_2\right] I + \\ &+ \hat{\beta}_3(\rho_2) D_2 + \hat{\beta}_4(\rho_2)\,\mathrm{grad}\,\rho_2 \otimes \mathrm{grad}\,\rho_2 \end{aligned} \tag{3.3}$$

where denotes the scalar product of two vectors and $\otimes$ denotes one diadic product of two vectors. The spherical part of the stress in Eq.(3.3) can be interpreted as the solid pressure p_s. The material moduli $\hat{\beta}_1$ and $\hat{\beta}_4$ are material parameters that reflect the distribution of the granular particles, and $\hat{\beta}_0$ plays a role akin to pressure in a compressible fluid and is given by an equation of state. The material modulus $\hat{\beta}_2$ is a viscosity akin to the second coefficient of viscosity in a compressible fluid and $\hat{\beta}_3$ denotes the viscosity (i.e., the resistance of the material to flow) of the granular solids. Recently, Rajagopal and Massoudi (1990), and Rajagopal *et al.* (1994) have outlined an experimental/theoretical approach to determine these material moduli. Based on the available experimental measurements of Savage (1979), Savage and Sayed (1984), and Hanes and Inman (1985) and the computer simulations of Walton and Braun (1986a,b), it is clear that granular materials exhibit normal stress effects. The above model (Eq. (3.3)) is a simplified version of the model proposed by Rajagopal and Massoudi (1990) which predicts the possibility of both the normal stress differences. Furthermore, Boyle and Massoudi (1990), using Enskog's dense gas theory, have obtained explicit expressions for the material moduli $\hat{\beta}_0$ through $\hat{\beta}_4$.

A mixture stress tensor is defined as (Green and Naghdi, 1969):

$$T_m = T_1 + T_2 \tag{3.4}$$

where

$$T_1 = (1-\nu)T_f \qquad \text{and} \qquad T_2 = T_s, \tag{3.5}$$

so that the mixture stress tensor reduces to that of a pure fluid as $\nu \to 0$ and to that of a granular material as $\phi \to 0$.[1] T_2 may also be written as $T_2 = \nu\hat{T}_s$ where $\hat{T}_s$ may be thought of as representing the stress tensor for some (quite densely packed) reference configuration of the granular material. Note that our choice for the partial stresses are symmetric.

The mechanical interaction between the mixture components, f_1, is written as (Johnson *et al.*, 1990)

$$\begin{aligned} f_1 &= A_1 \operatorname{grad}\nu + A_2 F(\nu)(v_2 - v_1) + A_3\nu\left(2trD_1^2\right)^{-\frac{1}{4}} D_1(v_2 - v_1) + \\ &+ A_4\nu(W_2 - W_1)(v_2 - v_1) + A_5 a_{vm} \end{aligned} \tag{3.6}$$

where a_{vm} is a properly frame invariant measure of the relative acceleration between the mixture components and $F(\nu)$ represents the dependence of the drag coefficient on the volume fraction. The terms in Eq.(3.7) reflect the presence of density gradients[2], drag, "slip-shear" lift, "spin" lift, and virtual mass, respectively. Müller's

[1] Note that $\phi \to 0$ is equivalent to $\nu \to 1$ only in a saturated mixture. Thus the theory allows for the mixture tending to a pure granular material without $\nu \to 1$ but to some value ν_m strictly less than unity, usually referred to as the maximum packing fraction. We are interested here, however, in the case when there is a sufficient amount of both the constituents and hence we are not close to either of the limiting cases. Further, in keeping with the usual weighting procedures in multiphase flow we shall represent T_m as

$$T_m = (1-\nu)T_f + \nu\hat{T}_s, \tag{3.7}$$

where $\hat{T}_s$ is discussed above.

[2] The actual form of this interaction should include the terms $\alpha_1 \operatorname{grad}\rho_1 + \alpha_2 \operatorname{grad}\rho_2$ where

(1968) work indicates that a term of the form A_1 grad v must be included in the interactions in order to get well-posed problems. The term multiplying A_3 is a generalization of Saffman's (1965, 1968) single particle result first proposed in this form by McTigue *et al.* (1986). One of the earliest studies examining the effect of lift force is discussed by Soo (1969) and Soo and Tung (1972); Massoudi (2001) discusses the importance of lift forces in multiphase flow analysis and Massoudi and Halow (2001) present the effect of lift forces in a Couette flow of a mixture.

4. Averaging

4.1. Conservation of mass

Consider a box shaped fixed region R in three-dimensional Euclidean space of volume V bounded by a surface ∂R of a. The box has unit depth in the z direction and surface $\partial R_t, \partial R_b, \partial R_1$ and ∂R_2 as indicated in Fig.1. All equations are postulated at the current time t and all field quantities are functions of x and t.

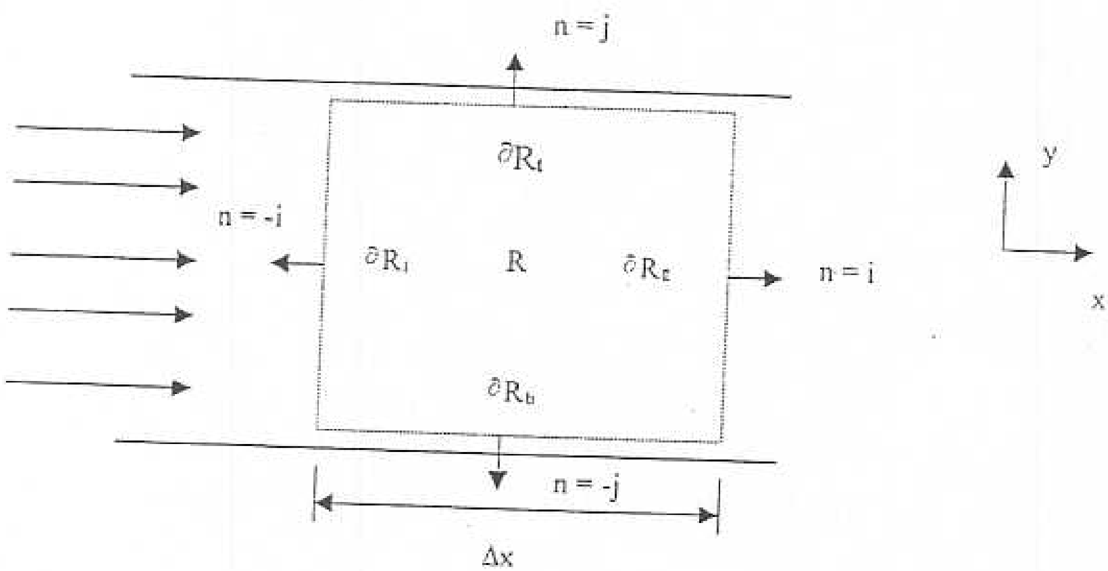

Fig.1. Control volume.

α_1 and α_2 are constants. If we assume that the system is a saturated mixture with incompressible components, this expression simplifies to A_1 grad v where $A_1 = \alpha_2 - \alpha_1$. Since no information concerning the coefficients α_1 and α_2 is available and a term of the same form arises in the balance of linear momentum in the granular solid stress tensor, this term will be neglected in the present work.

The appropriate balance of mass for a control volume is given by

$$\frac{\partial}{\partial t}\int_{R}\rho_{\alpha}dV=-\int_{\partial R}\rho_{\alpha}(\boldsymbol{v}_{\alpha}\cdot\boldsymbol{n})da+\int_{R}c_{\alpha}dV \tag{4.1}$$

where ρ_α is the density of the $\alpha-th$ constituent, $\boldsymbol{v}_\alpha$ is the velocity, $\boldsymbol{n}$ is the unit normal to the surface and c_α is the mass supply. Suppose there is no chemical reaction, i.e., $c_\alpha = 0$ and that the flow is steady and unidirectional, i.e., $\boldsymbol{v}_\alpha = v_\alpha(x,y,z)\boldsymbol{i}$. Equation (4.1) implies that

$$\int_{\partial R}\rho_{\alpha}(\boldsymbol{v}_{\alpha}\cdot\boldsymbol{n})\,da=0. \tag{4.2}$$

Denoting the entrance boundary as ∂R_1 and the exit as ∂R_2 we find

$$-\int_{\partial R_1}\rho_{\alpha}v_{\alpha}da+\int_{\partial R_2}\rho_{\alpha}v_{\alpha}da=0. \tag{4.3}$$

Let ∂R_1 be located at x and ∂R_2 be located at $x+\Delta x$. Also let us suppose that the cross-sectional area is constant. Then (by the mean value theorem[3])

$$\int_{\partial R_1}\rho_{\alpha}v_{\alpha}da=\rho_{\alpha}^{*}v_{\alpha}^{*}A \tag{4.4}$$

where the asterisks represent some average value on ∂R_1. Using Taylor's expansion

$$\int_{\partial R_2}\rho_{\alpha}v_{\alpha}da\approx\rho_{\alpha}^{*}v_{\alpha}^{*}A+\frac{d}{dx}\left(\rho_{\alpha}^{*}v_{\alpha}^{*}A\right)\Delta x. \tag{4.5}$$

Thus Eq.(4.3) implies

$$\frac{d}{dx}\left(\rho_{\alpha}^{*}v_{\alpha}^{*}A\right)=0. \tag{4.6}$$

[3]Mean value theorem for integrals: if f is continuous on $[a, b]$ then there is a number c in $[a, b]$ such that

$$\int_{a}^{b}f(x)dx=(b-a)f(c)$$

4.2. Balance of linear momentum for the fluid

The balance of linear momentum for the fluid is, in general

$$\frac{\partial}{\partial t}\int_R \rho_1 v_1 dV = -\int_{\partial R} \rho_1 v_1 (v_1 \cdot n) da + \\ + \int_{\partial R} T_1^T n \, da + \int_R (\rho_1 b_1 + f_1 + c_1 v_1) dV \tag{4.7}$$

where R represents the control volume, ∂R represents the surface of that control volume, ρ is density, v_1 is velocity, b_1 is the body force, f_1 is the interaction between the components, T_1 is the stress tensor, c_1 is the mass supply, and n is the unit normal to the surface of the control volume.
The subscript 1 refers to the fluid component. With the assumptions of steady flow and no chemical reaction the above equation reduces to

$$\int_{\partial R} \rho_1 v_1 (v_1 \cdot n) da = \int_{\partial R} T_1^T n \, da + \int_R (\rho_1 b + f_1) dV . \tag{4.8}$$

We now derive the averaged form of Eq.(4.8) by considering each term individually.

Convective term
The velocity fields of the fluid and solid are assumed to have the form:

$$v_1 = v(x, y) i, \quad \text{and} \qquad v_2 = u(x, y) i, \tag{4.9}$$

then

$$\int_{\partial R} \rho_1 v_1 (v_1 \cdot n) da = -\int_{\partial R_1} \rho_1 v^2 i da + \int_{\partial R_2} \rho_1 v^2 i da . \tag{4.10}$$

The Mean Value Theorem implies that

$$\int_{\partial R_1} \rho_1 v^2 i da = \rho_1^* (v^*)^2 A i \tag{4.11}$$

where the asterisks denote average values (i.e., averaged over the cross section) and A is the cross sectional area of the control volume. Assuming that Δx is small and applying Taylor's expansion yields

$$\int_{\partial R_2} \rho_1 v^2 \boldsymbol{i}\, da \approx \left[\rho_1^* \left(v^*\right)^2 A + \frac{d}{dx}\left[\rho_1^* \left(v^*\right)^2 A\right]\Delta x\right]\boldsymbol{i}. \tag{4.12}$$

Combining Eqs.(4.10), (4.11) and (4.12) gives the following result

$$\int_{\partial R} \rho \boldsymbol{v}_1 \left(\boldsymbol{v}_1 \cdot \boldsymbol{n}\right) da = \frac{d}{dx}\left[\rho_1^* \left(v^*\right)^2 A\right]\Delta x\, \boldsymbol{i}. \tag{4.13}$$

Body forces

The effect of the body force can be averaged over the control volume to yield

$$\int_R \rho_1 \boldsymbol{b}_1 dV = \rho_1^* \boldsymbol{b}_1 A \Delta x. \tag{4.14}$$

Interactions

In general, the interactions between the mixture components will include density gradients, drag, lift, and virtual mass. We shall neglect lift and density gradients but consider interactions due to drag and virtual mass as they are often more important for the flow of slurries. Thus the interaction term has the form

$$\boldsymbol{f}_1 = \alpha_3 (u - v) + \alpha_6 \boldsymbol{a}_{vm} \tag{4.15}$$

where $\boldsymbol{a}_{vm}$ is a frame-indifferent relative acceleration given by

$$\begin{aligned} \boldsymbol{a}_{vm} = & \left[\frac{D\boldsymbol{v}_2}{Dt} - (\text{grad}\, \boldsymbol{v}_2)(\boldsymbol{v}_2 - \boldsymbol{v}_1)\right] + \\ & - \left[\frac{D\boldsymbol{v}_1}{Dt} - (\text{grad}\, \boldsymbol{v}_1)(\boldsymbol{v}_1 - \boldsymbol{v}_2)\right]. \end{aligned} \tag{4.16}$$

With our assumptions about the velocity fields the interaction force vector becomes

$$\boldsymbol{f}_1 = \begin{pmatrix} \alpha_3 (u - v) + \alpha_6 \left(v \dfrac{\partial u}{\partial x} - u \dfrac{\partial v}{\partial x} \right) \\ 0 \\ 0 \end{pmatrix} \tag{4.17}$$

and

$$\int_R f_I dV = \left[\alpha_3^*\left(u^* - v^*\right) + \alpha_6^*\left(v^*\frac{\partial u^*}{\partial x} - u^*\frac{\partial v}{\partial x}\right)\right]A\Delta x i. \quad (4.18)$$

Stress tensor

The fluid stress tensor is given by

$$T_I = \left[-p_f + \lambda\, tr D_I\right] I + 2\mu D_I. \quad (4.19)$$

Following the same procedure as above, we have

$$\begin{aligned}\int_{\partial R}\left[-p_f + \lambda\, tr D_I\right] I n da = &-\int_{\partial R_I}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] i da + \\ &+\int_{\partial R_2}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] i da + \int_{\partial R_t}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] j da + \\ &-\int_{\partial R_b}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] j da\end{aligned} \quad (4.20)$$

where

$$\int_{\partial R_I}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] da = \left[-p_f^* + \lambda^*\frac{\partial v^*}{\partial x}\right] A \quad (4.21)$$

and

$$\int_{\partial R_2}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] da = \left[-p_f^* + \lambda^*\frac{\partial v^*}{\partial x}\right] A + \frac{d}{dx}\left[-p_f^* + \lambda^*\frac{\partial v^*}{\partial x}\right] A\Delta x, \quad (4.22)$$

so that

$$\begin{aligned}\int_{\partial R}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] I n da = &\frac{d}{dx}\left[-p_f^* + \lambda^*\frac{\partial v^*}{\partial x}\right] A\Delta x i + \\ &+\int_{\partial R_t}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] j da - \int_{\partial R_b}\left[-p_f + \lambda\frac{\partial v}{\partial x}\right] j da.\end{aligned} \quad (4.23)$$

Next,

$$\int_{\partial R} 2\mu\, D_{I} n\, da = -\int_{\partial R_1} 2\mu \frac{\partial v}{\partial x} i da + \int_{\partial R_2} 2\mu \frac{\partial v}{\partial x} i da - \int_{\partial R_b} \mu \frac{\partial v}{\partial y} i da + \\ + \int_{\partial R_t} \mu \frac{\partial v}{\partial x} i da - \int_{\partial R_1} \mu \frac{\partial v}{\partial y} j da + \int_{\partial R_2} \mu \frac{\partial v}{\partial y} j da, \tag{4.24}$$

$$\int_{\partial R_1} 2\mu \frac{\partial v}{\partial x} da = 2\mu^* \frac{\partial v^*}{\partial x} A, \tag{4.25}$$

$$\int_{\partial R_2} 2\mu \frac{\partial v}{\partial x} da = 2\mu^* \frac{\partial v^*}{\partial x} A + \frac{d}{dx}\left[2\mu^* \frac{\partial u^*}{\partial x} A\right]\Delta x. \tag{4.26}$$

The shear force of the fluid on the upper and lower surfaces of the control volume is defined as

$$F_w = \tau_f \Delta x = \int_{\partial R_t} \mu \frac{\partial v}{\partial y} da - \int_{\partial R_t} \mu \frac{\partial v}{\partial y} da \tag{4.27}$$

and

$$\int_{\partial R} 2\mu D_I n da = \frac{d}{dx}\left[2\mu^* \frac{dv^*}{dx} A\right]\Delta x i + \\ -\tau_f \Delta x i - \int_{\partial R_1} \mu \frac{\partial v}{\partial y} j da + \int_{\partial R_2} \mu \frac{\partial v}{\partial y} j da. \tag{4.28}$$

Thus the balance of linear momentum for the fluid phase in the x-direction becomes

$$\frac{d}{dx}\left[\rho_I^* (v^*)^2 A\right]\Delta x = \frac{d}{dx}\left[-p_f^* + \lambda^* \frac{\partial v^*}{\partial x}\right] A\Delta x + \frac{d}{dx}\left[2\mu^* \frac{dv^*}{dx} A\right]\Delta x + \\ -\tau_f \Delta x + \rho_I^* (b_I)_x A\Delta x + \left[\alpha_3^* (u^* - v^*) + \alpha_6^* \left(v^* \frac{\partial u^*}{\partial x} - u^* \frac{\partial v^*}{\partial x}\right)\right] A\Delta x \tag{4.29}$$

and in the y-direction, we have

$$\int_{\partial R_t}\left[-p_f + \lambda \frac{\partial v}{\partial x}\right] da - \int_{\partial R_b}\left[-p_f + \lambda \frac{\partial v}{\partial x}\right] da - \int_{\partial R_I} \mu \frac{\partial v}{\partial y} da + \\ + \int_{\partial R_2} \mu \frac{\partial v}{\partial y} da + \rho_I^* (b_I)_y A\Delta x = 0. \tag{4.30}$$

4.3. Balance of linear momentum for the granular materials

The balance of linear momentum for the granular materials is

$$\begin{aligned}\frac{\partial}{\partial t}\int_R \rho_2 v_2 dV = -\int_{\partial R} \rho_2 v_2 (v_2 \cdot n) da + \\ + \int_{\partial R} T_2^T n da + \int_R (\rho_2 b_2 - f_I + c_2 v_2) dV,\end{aligned} \tag{4.31}$$

where R represents the control volume, ∂R represents the surface of that control volume, ρ_2 is density, v_2 is velocity, b_2 is the body force, f_I is the interaction between the components, T_2 is the stress tensor, c_2 is the mass supply, and n is the unit normal to the surface of the control volume. The subscript 2 refers to the solid component. With the assumptions of steady flow and no chemical reaction the above equation reduces to

$$\int_{\partial R} \rho_2 v_2 (v_2 \cdot n) da = \int_{\partial R} T_2^T n da + \int_R (\rho_2 b_2 - f_I) dV . \tag{4.32}$$

The solid stress tensor is given by

$$T_2 = [-p_s + \hat{\beta}_2 tr D_2] I + \hat{\beta}_3 D_2 + \hat{\beta}_4 \text{grad} \nu \otimes \text{grad} \nu \tag{4.33}$$

where

$$p_s = -\hat{\beta}_0 - \hat{\beta}_1 \text{grad } \nu \cdot \text{grad } \nu . \tag{4.34}$$

The following results are analogous to those for the fluid

$$\int_{\partial R} \rho v_2 (v_2 \cdot n) da = \frac{d}{dx}\left[\rho_2^* (u^*)^2 A\right] \Delta x i, \tag{4.35}$$

$$\int_R \rho_2 b dV = \rho_2^* b A \Delta x, \tag{4.36}$$

$$\int_R f_I dV = \left[\alpha_3^* (u^* - v^*) + \alpha_6^* \left(v^* \frac{\partial u^*}{\partial x} - u^* \frac{\partial v^*}{\partial x} \right)\right] A \Delta x i, \tag{4.37}$$

$$\int_{\partial R}\left[-p_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]Inda=\frac{d}{dx}\left[-p_s^*+\hat{\beta}_2^*\frac{\partial u}{\partial x}\right]A\Delta xi+ \\ +\int_{\partial R_t}\left[-p_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]jda-\int_{\partial R_b}\left[-p_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]jda, \tag{4.38}$$

$$\int_{\partial R}\hat{\beta}_3 D_2 nda=\frac{d}{dx}\left[\hat{\beta}_3^*\frac{du^*}{dx}A\right]\Delta xi-\tau_s\Delta xi+ \\ -\frac{1}{2}\int_{\partial R_1}\hat{\beta}_3\frac{\partial u}{\partial y}jda+\frac{1}{2}\int_{\partial R_2}\hat{\beta}_3\frac{\partial u}{\partial y}jda \tag{4.39}$$

where τ_s is defined in an identical fashion to τ_f in Eq.(4.27) through

$$\tau_s\Delta x=\int_{\partial R_t}\hat{\beta}_3\frac{\partial u}{\partial y}da-\int_{\partial R_b}\hat{\beta}_3\frac{\partial u}{\partial y}da. \tag{4.40}$$

The solid stress tensor has an additional term not found in the fluid equation. To simplify the following calculations, we define

$$M=\mathrm{grad}\nu\otimes\mathrm{grad}\nu, \tag{4.41}$$

which for our assumed form of the density field becomes

$$M=\begin{pmatrix}\left(\frac{\partial\nu}{\partial x}\right)^2 & \frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y} & 0\\ \frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y} & \left(\frac{\partial\nu}{\partial y}\right)^2 & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{4.42}$$

The additional term in the solid balance of linear momentum is

$$\int_{\partial R}\hat{\beta}_4 Mnda=-\int_{\partial R_1}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2 ida+\int_{\partial R_2}\hat{\beta}_4\left(\frac{\partial\nu}{\partial x}\right)^2 ida+ \\ +\int_{\partial R_t}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}ida-\int_{\partial R_b}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}ida+\int_{\partial R_2}\hat{\beta}_4\frac{\partial\nu}{\partial x}\frac{\partial\nu}{\partial y}ida+ \tag{4.43}$$

$$-\int_{\partial R_1}\hat{\beta}_4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}jda+\int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2 jda-\int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2 jda. \qquad \text{cont.(4.43)}$$

Proceeding as before, we have

$$\int_{\partial R_1}\hat{\beta}_4\left(\frac{\partial v}{\partial x}\right)^2 da=\hat{\beta}_4^*\left(\frac{\partial v^*}{\partial x}\right)^2 A, \tag{4.44}$$

$$\int_{\partial R_2}\hat{\beta}_4\left(\frac{\partial v}{\partial x}\right)^2 da=\hat{\beta}_4^*\left(\frac{\partial v^*}{\partial x}\right)^2 A+\frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial v^*}{\partial x}\right)^2 A\right]\Delta x. \tag{4.45}$$

The shear force of the solid on the upper and lower surfaces of the control volume due to the normal stresses is defined as

$$\tau_n\Delta x=\int_{\partial R_t}\hat{\beta}_4\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}da-\int_{\partial R_b}\hat{\beta}_4\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}da \tag{4.46}$$

and

$$\begin{aligned}\int_{\partial R}\hat{\beta}_4 Mnda&=\frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial v^*}{\partial x}\right)^2 A\right]\Delta x i-\tau_n\,\Delta x i+\\&+\int_{\partial R_2}\hat{\beta}_4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}jda-\int_{\partial R_1}\hat{\beta}_4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}jda+\\&+\int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2 jda-\int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2 jda.\end{aligned} \tag{4.47}$$

Thus the balance of linear momentum in the x-direction becomes

$$\begin{aligned}\frac{d}{dx}\left[\rho_2^*\left(u^*\right)^2 A\right]\Delta x&=\frac{d}{dx}\left[-\rho_s^*+\hat{\beta}_2^*\frac{\partial u^*}{\partial x}\right]A\Delta x+\\&+\frac{d}{dx}\left[\hat{\beta}_3^*\frac{\partial u^*}{\partial x}A\right]\Delta x-\tau_s\Delta x+\frac{d}{dx}\left[\hat{\beta}_4^*\left(\frac{\partial v^*}{\partial x}\right)^2 A\right]\Delta x-\tau_n\Delta x+\\&+\rho_2^* b_x A\Delta x-\left[\alpha_3^*\left(u^*-v^*\right)+\alpha_6^*\left(v^*\frac{\partial u^*}{\partial x}-u^*\frac{\partial v^*}{\partial x}\right)\right]A\Delta x\end{aligned} \tag{4.48}$$

and in the y-direction, we have

$$\int_{\partial R_t}\left[-\rho_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]da-\int_{\partial R_b}\left[-\rho_s+\hat{\beta}_2\frac{\partial u}{\partial x}\right]da-\frac{1}{2}\int_{\partial R_1}\hat{\beta}_3\frac{\partial u}{\partial y}da+$$
$$+\frac{1}{2}\int_{\partial R_2}\hat{\beta}_3\frac{\partial u}{\partial y}da-\int_{\partial R_1}\hat{\beta}_4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}da+\int_{\partial R_2}\hat{\beta}_4\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}da+ \tag{4.49}$$
$$+\int_{\partial R_t}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2da-\int_{\partial R_b}\hat{\beta}_4\left(\frac{\partial v}{\partial y}\right)^2da+\rho_2^*b_yA\Delta x=0.$$

5. Summary and conclusions

Combining Eqs.(4.13), (4.14), (4.18), (4.23), (4.28) with Eqs.(4.8) and (4.35)÷(4.39), and Eq.(4.47) with Eq.(4.32), and treating the averaged values for the variable as the variable, yields (in the x-direction)

$$\frac{d}{dx}\left(\rho_1^*v^{*2}\right)=-\frac{dp_f^*}{dx}+\left(\lambda^*+2\mu^*\right)\frac{d^2v^*}{dx^2}-\frac{1}{A}\tau_f+\rho_1^*b_x+$$
$$+\alpha_3^*\left(u^*-v^*\right)+\alpha_6^*\left[v^*\frac{du^*}{dx}-u^*\frac{dv^*}{dx}\right], \tag{5.1}$$

$$\frac{d}{dx}\left(\rho_2^*u^{*2}\right)=-\frac{dp_s^*}{dx}+\left(\hat{\beta}_2^*+\hat{\beta}_3^*\right)\frac{d^2u^*}{dx^2}+$$
$$-\frac{1}{A}\tau_s+\rho_2^*b_x+2\hat{\beta}_4^*\frac{dv^*}{dx}\frac{d^2v^*}{dx^2}-\frac{1}{A}\tau_n+ \tag{5.2}$$
$$-\alpha_3^*\left(u^*-v^*\right)+\alpha_6^*\left[v^*\frac{du^*}{dx}-u^*\frac{dv^*}{dx}\right]$$

where b_x is the x-component of the body force and

$$\tau_f=\frac{2\mu}{\Delta x}\int_{\partial R_t}\frac{\partial v}{\partial y}da, \tag{5.3}$$

$$\tau_n=\frac{2\hat{\beta}_4}{\Delta x}\int_{\partial R_t}\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}da, \tag{5.4}$$

$$\tau_s = \frac{\hat{\beta}_3}{\Delta x}\int_{\partial R_t} \frac{\partial u}{\partial y} da . \tag{5.5}$$

In many applications (Soo, 1967), the equation of interest is the fluid-particle (or mixture) momentum equation. This equation is obtained by adding Eqs.(5.1) and (5.2)

$$\begin{aligned} &\frac{d}{dx}\left(\rho_1^* v^{*2}\right) + \frac{d}{dx}\left(\rho_2^* u^{*2}\right) = -\frac{dp_f^*}{dx} - \frac{dp_s^*}{dx} + \\ &+\left(\lambda^* + 2\mu^*\right)\frac{d^2 v^*}{dx^2} + \left(\hat{\beta}_2^* + \hat{\beta}_3^*\right)\frac{d^2 u^*}{dx^2} - \frac{1}{A}\tau_f - \frac{1}{A}\tau_s + \\ &+\rho_1^* b_x + \rho_2^* b_x + 2\hat{\beta}_4^* \frac{dv^*}{dx}\frac{d^2 v^*}{dx^2} - \frac{1}{A}\tau_n , \end{aligned} \tag{5.6}$$

which can be rewritten as

$$\begin{aligned} &\frac{d}{dx}\left(\rho_1^* v^{*2}\right) + \frac{d}{dx}\left(\rho_2^* u^{*2}\right) = -\frac{d}{dx}\left(p_f^* + p_s^*\right) + \\ &+\left(\lambda^* + 2\mu^*\right)\frac{d^2 v^*}{dx^2} + \left(\hat{\beta}_2^* + \hat{\beta}_3^*\right)\frac{d^2 u^*}{dx^2} - \frac{1}{A}\left(\tau_f + \tau_s\right) + \\ &+\left(\rho_1^* + \rho_2^*\right) b_x + 2\hat{\beta}_4^* \frac{dv^*}{dx}\frac{d^2 v^*}{dx^2} - \frac{1}{A}\tau_n . \end{aligned} \tag{5.7}$$

In the spirit of previous work in this area, the balance of momentum in the y-direction is not included in the averaged equations. Of course, the terms in the y-direction are such that the pressure field adjusts itself to satisfy the governing equations. Soo (1967, p.279, Eq.(7.7)) presents the overall momentum equation for a mixture composed of a Newtonian fluid and a dilute phase consisting of solid particles

$$\rho u \frac{du}{dx} + \sum_s \rho_p^s u_p^s \frac{du_p^s}{dx} = -\frac{dP}{dx} - \frac{\tau_w \pi}{A} \tag{5.8}$$

where π is the perimeter of the duct with area A and τ_w is given by

$$C_f = \tau_w \Big/ \frac{\rho u^2}{2} . \tag{5.9}$$

Obviously, the fact that particles are not interacting with each other makes Eq.(5.8) different from our equation for the mixture, Eq.(5.7). However, we can see that if we set $\beta_2 = \beta_3 = 0$ (i.e., no shear or bulk viscosity for the solid phase), and $\beta_4 = 0$ (i.e., no particle distribution factor), then $\tau_s = \tau_n = 0$, and Eq.(5.7) reduces to

$$\frac{d}{dx}\left(\rho_1^* v^{*2}\right) + \frac{d}{dx}\left(\rho_2^* u^{*2}\right) = -\frac{dP}{dx} + \\ + \left(\lambda^* + 2\mu^*\right)\frac{d^2 v^*}{dx^2} - \frac{\tau_f}{A} + \left(\rho_1^* + \rho_2^*\right) b_x \tag{5.10}$$

where we have defined $P = p_f^* + p_s^*$. Therefore, it is clear that if the mixture is treated as a dense suspension, where the particle phase is also treated as a continuum phase given by an appropriate constitutive equation for the stress tensor, then the mixture momentum equation will contain additional terms due to the viscosity and normal stress coefficients.

Acknowledgments

I would like to thank Professor K.R. Rajagopal for many suggestions and comments which have led to improvements in this paper. This problem was suggested and started by him, and he should really be the primary author. I am indebted to him. I also would like to thank Mrs. Diana Mason of the Word Processing Center at NETL-Pittsburgh for typing this manuscript.

Nomenclature

- a – acceleration vector
- a_{vm} – relative acceleration between components
- A_i – interaction coefficients, $i = 1$ to 5
- b – body force vector
- D – symmetric part of the velocity gradient
- f_I – interaction force vector
- F – deformation gradient
- F – volume fraction dependence of drag
- g – gravitational acceleration
- I – identity tensor
- L – gradient of velocity vector
- p – fluid pressure
- Re – Reynolds number
- T – stress tensor

v – velocity vector
W – spin tensor
x – position vector
β_i – granular solid coefficients $(i = 0-4)$
κ – mapping function
φ – volume fraction of fluid
λ_f – second coefficient of fluid viscosity
μ – first coefficient of fluid viscosity
ν – volume fraction of the solid
ρ – density

Subscripts

$1, f$ – referring to the fluid phase
$2, s$ – referring to the solid phase
m – referring to the mixture

Superscripts

T – transpose
$*$ – dimensionless quantity

Other symbols

$\nabla \bullet$ – divergence operator
∇ – gradient operator
tr – trace of a tensor
$\otimes$ – outer product
$\bullet$ – dot product

References

Adkins J.E. (1963a): *Non-linear diffusion, 1. Diffusion and flow of mixture of fluids*. – Philos. Trans. R. Soc., London, vol.255A, pp.607-633.

Adkins J.E. (1963b): *Non-linear diffusion, 2. Constitutive equations for mixture of isotropic fluids*. – Philos. Trans. R. Soc., London, vol.255A, pp.635-648.

Anderson T.B. and Jackson R. (1967): *A fluid mechanical descriptions of fluidized beds, 1. Equations of motion*. – Ind. Eng. Chem. Fund., vol.6, pp.527-539.

Atkin R.J. and Craine R.E. (1976): *Continuum theories of mixtures: basic theory and historical development*. – Quart. J. Mech. Appl. Math., vol.29, pp.208-244.

Bedford A. and Drumheller D.S. (1983): *Recent advances: theories of immiscible and* *structured mixtures.* – Int. J. Engng. Sci., vol.21, pp.863-960.

Bowen R.M. (1976): *Theory of mixtures*, In: Continuum Physics (A.C. Eringen, Ed.). New York : Academic Press, vol.3, pp.1-127.

Boyle E.J. and Massoudi M. (1990): *A theory for granular materials exhibiting normal stress effects based on Enskog's dense gas theory.* – Int. J. Engng. Sci., vol.28, pp.1261-1275.

Drew D.A. (1983): *Mathematical modeling of two phase flow.* – Annual Rev. Fluid Mech., vol.15, pp.261-291.

Drew D.A. and Segel L.A. (1971): *Averaged equations for two-phase flows.* – Studies in Applied Math., vol.L, pp.205-231.

Fick A.(1855): *Uber diffusion.* – Ann. der Physik, vol.94, pp.59-86.

Goodman M.A. and Cowin S.C. (1972): *A continuum theory for granular materials.* – Arch. Rat. Mech. Anal., vol.44, pp.249-266.

Goodman M.A. and Cowin S.C. (1971): *Two problems in gravity flow of granular materials.* – J. Fluid Mech., vol.45, pp.321-339.

Green A.E. and Naghdi P.M. (1969): *On basic equations for mixtures.* – Q. J. Mech. Appl. Math, vol.22, pp.427-438.

Hanes D.M. and Inman D.L. (1985): *Observations of rapidly flowing granular-fluid materials.* – J. Fluid Mech., vol.150, pp.357-380.

Hestroni G. (1982): *Handbook of Multiphase Systems.* – New York: Hemisphere Publishing Corp.

Johnson G., Massoudi M. and Rajagopal K.R. (1990): *A Review of Interaction Mechanisms in Fluid-Solid Flows.* – Technical Report, U.S. Dept. of Energy, DOE/PETC/TR-90/9.

Johnson G., Massoudi M. and Rajagopal K.R. (1991a): *Flow of a fluid-solid mixture between parallel plates.* – Chem. Engng. Sci., vol.46, pp.1713-1723.

Johnson G., Massoudi M. and Rajagopal K.R. (1991b): *Flow of a fluid infused with solid particles through a pipe.* – Int. J. Engng. Sci., vol.29, pp.649-661.

Marcus R.D., Lang L.S., Klinzing G.E. and Rizk F. (1990): *Pneumatic Conveying of Solids.* – New York: Chapman and Hall.

Massoudi M. (2001): *On the importance of material frame-indifference and lift forces in multiphase flows.* – Submitted for publication.

Massoudi M. and Halow J.S. (2001): *The effect of lift forces in Couette flow of mixture.* – Submitted for publication.

Massoudi M. and Johnson G. (2000): *On the flow of a fluid-particle mixture between two rotating cylinders, suing the theory of interacting continua.* – Int. J. Non-Linear Mech., vol.35, pp.1045-1058.

Massoudi M. and Phuoc T.X. (2000): *The effect of slip boundary condition on the flow of* *granular materials; A continuum approach.* – Int. J. Non-Linear Mech., vol.35, pp.745-761.

Massoudi M., Rajagopal K.R. and Phuoc T.X. (1999): *On the fully developed flow of a dense particulate mixture in a pipe.* – Powder Tech., vol.104, pp.258-268.

Müller I. (1968): *A thermodynamic theory of mixtures of fluids.* – Arch. Rat. Mech. Anal., vol.28, pp.1-39.

McTigue D.F., Givler R.C. and Nunziato J.W. (1986): *Rheological effects of nonuniform particle distributions in dilute suspensions.* – J. Rheology, vol.30, pp.1053-1076.

Rajagopal K.R. and Massoudi M. (1990): *A Method for Measuring the Material Moduli of Granular Materials: Flow in an Orthogonal Rheometer.* – Technical Report, U.S. Dept. of Energy DOE/PETC/TR-90/3.

Rajagopal K.R. and Tao L. (1995): *Mechanics of Mixtures.* – World Scientific, River Edge, NJ.

Rajagopal K.R., Massoudi M. and Ekmann J.M. (1990): *Mathematical modeling of fluid-solid mixtures,* In: Recent Developments in Structured Continua II (D. DeKee and P.N. Kaloni, Eds.). Longman, pp.236-248.

Rajagopal K.R., Massoudi M. and Wineman A.S. (1994): *Flow of granular materials between rotating disks.* – Mech. Res. Comm., vol.21, pp.629-634.

Rajagopal K.R., Wineman A.S. and Gandhi M. (1986): *On boundary conditions for a certain class of problems in mixture theory.* – Int. J. Engng. Sci., vol.24, pp.1453-1463.

Saffman P.G. (1965): *The lift on a small sphere in slow shear flow.* – J. Fluid Mech., vol.22, pp.385-400.

Saffman P.G. (1968): *Corrigendum.* – J. Fluid Mech., vol.31, pp.624.

Savage S.B. (1979): *Gravity flow of cohesionless granular materials in chutes and channels.* – J. Fluid Mech., vol.92, pp.53-96.

Savage S.B. and Sayed M. (1984): *Stresses developed by dry cohesionless granular materials sheared in an annular shear cell.* – J. Fluid Mech., vol.142, pp.391-430.

Shi J.J.J., Rajagopal K.R. and Wineman A.S. (1981): *Applications of the theory of interacting continua to the diffusion of a fluid through a non-linear elastic media.* – Int. J. Engng. Sci., vol.19, pp.871-889.

Soo S.L. (1967): *Fluid Dynamics of Multi-Phase Systems.* – Blaisdell Publishing Co., Waltham, MA.

Soo S.L. (1969): *Pipe flow of suspensions.* – Appl. Sci. Res., vol.21, pp.68-84.

Soo S.L. (1984): *Development of theories on liquid-solid flows.* – J. Pipelines, vol.4, pp.137-142.

Soo S.L. (1987): *Pipe flow of a dense suspension.* – J. Pipelines, vol.6. pp.193-203.

Soo S.L. (1989): *Particulates and Continuum.* – New York: Hemisphere Publishing Corp.

Soo S.L. (1990a): *Challenges in the fundamentals of multiphase flow.* – The Third International Lecture Course on Multiphase Flow, Osaka, Japan, pp.24-45.

Soo S.L. (1990b): *Multiphase Fluid Dynamics.* – Brookfield, Vermont: Science Press and Gower Technical.

Soo S.L. (1991): *Comparisons of formulations of multiphase flow.* – Powder Tech., vol.66, pp.1-7.

Soo S.L. and Tung S.K. (1972): *Deposition and entrainment in pipe flow of a suspension.* – Powder Tech., vol.6, p.283.

Tao L. and Rajagopal K.R. (1995): *On the boundary conditions in mixture theory.* In: Recent Advances in Elasticity, Viscoelasticity, and Inelasticity (K.R. Rajagopal, Ed.). – World Scientific, Singapore.

Tao L. and Rajagopal K.R. (1999): *Flows of bubbly liquids.* – Appl. Mech. Engng., vol.4, pp.407-442.

Truesdell C. (1957): *Sulle basi della termomeccanica.* – Rend Lincei, Series 8, vol.22, pp.33-38 and pp.158-166.

Truesdell C. (1984): *Rational Thermodynamics.* – New York: Springer-Verlag, 2nd edition.

Wallis G.B. (1969): *One-dimensional two-phase flow.* – New York: McGraw-Hill.

Walton O.R. and Braun R.L. (1986a): *Stress calculations for assemblies of inelastic spheres in uniform shear.* – Acta Mech., vol.63, pp.73-86.

Walton O.R. and Braun R.L. (1986b): *Viscosity, granular-temperature, and stress calculations for shearing assemblies of inelastic, frictional disks.* – J. Rheology, vol.30, pp.949-989.

Received: June 27, 2001

Revised: August 23, 2001